Wednesday, September 23, 2015

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Problem 1

Problem. Set up the definite integral that gives the area of the region

$$y_1 = x^2 - 6x,$$
$$y_2 = 0$$

Solution. The graphs intersect at $x = 0$ and $x = 6$ and y_2 is the uppermost function. So the integral is

$$\int_0^6 (-x^2 + 6x)\, dx.$$

Problem 3

Problem. Set up the definite integral that gives the area of the region

$$y_1 = x^2 - 4x + 3,$$
$$y_2 = -x^2 + 2x + 3$$

Solution. First, we must find where y_1 and y_2 intersect. Solve $y_1 = y_2$ for x. (It's obvious from the figure, but let's solve for x anyway.)

$$\begin{aligned} x^2 - 4x + 3 &= -x^2 + 2x + 3, \\ 2x^2 - 6x &= 0, \\ x^2 - 3x &= 0, \\ x(x-3) &= 0. \end{aligned}$$

Yep, the graphs intersect at $x = 0$ and $x = 3$ and y_2 is the uppermost function. So the integral is

$$\int_0^3 \big((-x^2 + 2x + 3) - (x^2 - 4x + 3)\big)\, dx = \int_0^3 (-2x^2 + 6x)\, dx.$$

Problem 5

Problem. Set up the definite integral that gives the area of the region

$$y_1 = 3(x^3 - x),$$
$$y_2 = 0$$

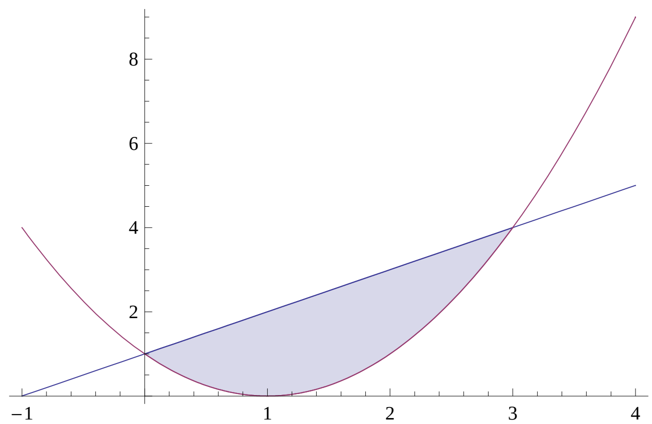
Solution. It is perfectly clear that the graphs intersect at $x = -1$, $x = 0$, and $x = 1$. However, y_1 is uppermost from -1 to 0 and y_2 is uppermost from 0 to 1. So we need two integrals.

$$\int_{-1}^{0} 3(x^3 - x)\, dx + \int_{0}^{1} (-3(x^3 - x))\, dx.$$

Problem 13

Problem. Determine which value best approximates the area of the region bounded by the graphics of $f(x) = x + 1$ and $g(x) = (x - 1)^2$.

Solution. Here is a “sketch:”



The region fits inside a rectangle of area 12 (3 units wide and 4 units high). It appears to occupy a bit less than half the area. So the best guess is (d) 4.

Problem 15

Problem. Find the area of the region by integrating (a) with respect to x and (b) with respect y.

$$x = 4 - y^2,$$
$$x = y - 2$$

Solution. Find where the graphs intersect.

$$\begin{aligned} 4 - y^2 &= y - 2, \\ -y^2 - y + 6 &= 0, \\ y^2 + y - 6 &= 0, \\ (y+3)(y-2) &= 0. \end{aligned}$$

So $y = -3$ or $y = 2$. These correspond to the points $(-5, -3)$ and $(0, 2)$. Also, the vertex of the parabola is at $(4, 0)$.

Integrate with respect to x

We must set up two integrals, one from $x = -5$ to $x = 0$ and the other from $x = 0$ to $x = 4$. We must also express the curves as functions of x, not y. The line is $y = x + 2$ and the parabola is in two parts. The upper part is $y = \sqrt{4 - x}$ and the lower part is $y = -\sqrt{4 - x}$.

$$\begin{aligned} \text{Area} &= \int_{-5}^{0} \left((x+2) - (-\sqrt{4-x})\right)\, dx + \int_{0}^{4} \left((\sqrt{4-x}) - (-\sqrt{4-x})\right)\, dx \\ &= \int_{-5}^{0} \left(x + 2 + \sqrt{4-x})\right)\, dx + 2\int_{0}^{4} \left(\sqrt{4-x}\right)\, dx. \end{aligned}$$

Let $u = 4 - x$ and $du = -dx$. Then $x = 4 - u$ and $dx = -du$ and

$$\begin{aligned} \text{Area} &= -\int_{9}^{4} \left((4-u) + 2 + \sqrt{u})\right)\, du - 2\int_{4}^{0} \left(\sqrt{u}\right)\, du \\ &= -\int_{9}^{4} \left(6 - u + \sqrt{u})\right)\, du - 2\int_{4}^{0} \left(\sqrt{u}\right)\, du \\ &= -\left[6u - \frac{1}{2}u^2 + \frac{2}{3}u^{3/2}\right]_9^4 - 2\left[\frac{2}{3}u^{3/2}\right]_4^0 \\ &= -\left(\left(24 - \frac{1}{2}\cdot 4^2 + \frac{2}{3}\cdot 4^{3/2}\right) - \left(54 - \frac{1}{2}\cdot 9^2 + \frac{2}{3}\cdot 9^{3/2}\right)\right) - 2\left(-\frac{2}{3}\cdot 4^{3/2}\right) \\ &= -\left(\frac{64}{3} - \frac{63}{2}\right) + \left(\frac{32}{3}\right) \\ &= \frac{125}{6}. \end{aligned}$$

Integrate with respect to y

Only one integral is necessary. y ranges from -3 to 2. The "upper" function is $x = 4 - y^2$ and the "lower" function is $x = y - 2$.

$$\begin{aligned}
\text{Area} &= \int_{-3}^{2} \big((4 - y^2) - (y - 2)\big)\ dy \\
&= \int_{-3}^{2} \big(-y^2 - y + 6\big)\ dy \\
&= \left[-\frac{1}{3}y^3 - \frac{1}{2}y^2 + 6y\right]_{-3}^{2} \\
&= \left(-\frac{1}{3}\cdot 2^3 - \frac{1}{2}\cdot 2^2 + 12\right) - \left(-\frac{1}{3}(-3)^3 - \frac{1}{2}(-3)^2 - 18\right) \\
&= \left(\frac{22}{3}\right) + \left(\frac{27}{2}\right) \\
&= \frac{125}{6}.
\end{aligned}$$

Problem 16

Problem. Find the area of the region by integrating (a) with respect to x and (b) with respect y.

$$y = x^2,$$
$$y = 6 - x$$

Solution. Find where the graphs intersect.

$$\begin{aligned}
x^2 &= 6 - x, \\
x^2 + x - 6 &= 0, \\
(x + 3)(x - 2) &= 0.
\end{aligned}$$

The graphs intersect at $(-3, 9)$ and $(2, 4)$.

Integrate with respect to x

Along the x-axis, there is only one upper function $(6-x)$ and only one lower function (x^2).

$$\begin{aligned}
\text{Area} &= \int_{-3}^{2} \left((6-x) - x^2\right)\ dx \\
&= \left[6x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-3}^{2} \\
&= \left(12 - \frac{1}{2}\cdot 2^2 - \frac{1}{3}\cdot 2^3\right) - \left(-18 - \frac{1}{2}(-3)^2 - \frac{1}{3}(-3)^3\right) \\
&= \left(\frac{22}{3}\right) + \left(\frac{27}{2}\right) \\
&= \frac{125}{6}.
\end{aligned}$$

Integrate with respect to y

Write the functions as functions of y.

$$\begin{aligned}
x &= 6 - y, \\
x &= \sqrt{y}, \\
x &= -\sqrt{y}.
\end{aligned}$$

We must set up two integrals along the y-axis. The first goes from 0 to 4 and the second goes from 4 to 9. (This sure sounds a lot like problem 15.)

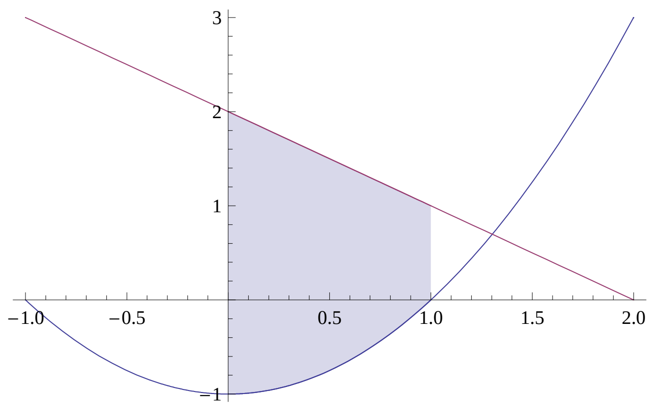
$$\begin{aligned}
\text{Area} &= \int_{0}^{4} \left(\sqrt{y} - (-\sqrt{y})\right)\ dy + \int_{4}^{9} \left((6-y) - (-\sqrt{y})\right)\ dy \\
&= 2\left[\frac{2}{3}y^{3/2}\right]_0^4 + \left[6y - \frac{1}{2}y^2 + \frac{2}{3}y^{3/2}\right]_4^9 \\
&= 2\left(\frac{2}{3}\cdot 4^{3/2}\right) + \left(\left(54 - \frac{1}{2}\cdot 9^2 + \frac{2}{3}\cdot 9^{3/2}\right) - \left(24 - \frac{1}{2}\cdot 4^2 + \frac{2}{3}\cdot 4^{3/2}\right)\right) \\
&= \left(\frac{32}{3}\right) + \left(\frac{63}{2} - \frac{64}{3}\right) \\
&= \frac{125}{6}.
\end{aligned}$$

Problem 17

Problem. Sketch the region bounded by the graphs of the equations and find the area of the region.

$$y = x^2 - 1,$$
$$y = -x + 2,$$
$$x = 0,$$
$$x = 1$$

Solution. If we sketch the graph, we see which is the upper function.



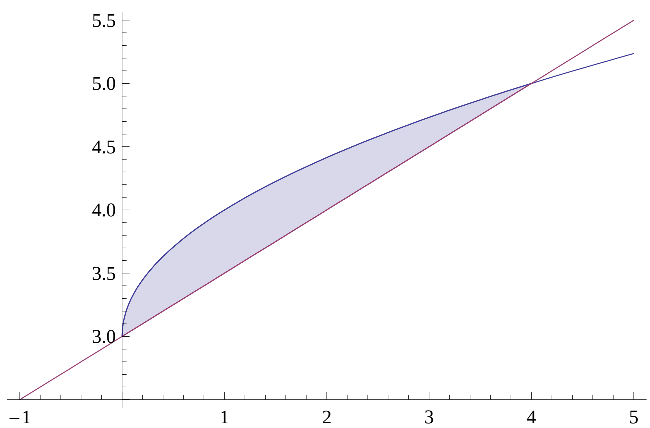
$$\begin{aligned}
\text{Area} &= \int_0^1 \big((-x+2) - (x^2-1)\big)\ dx \\
&= \int_0^1 \big(-x^2 - x + 3\big)\ dx \\
&= \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x\right]_0^1 \\
&= -\frac{1}{3} - \frac{1}{2} + 3 \\
&= \frac{13}{6}.
\end{aligned}$$

Problem 23

Problem. Sketch the region bounded by the graphs of the equations and find the area of the region.

$$f(x) = \sqrt{x} + 3,$$
$$g(x) = \frac{1}{2}x + 3$$

Solution. If we sketch the graph, we see which is the upper function.



We also find the intersection points to be $(0, 3)$ and $(4, 5)$.

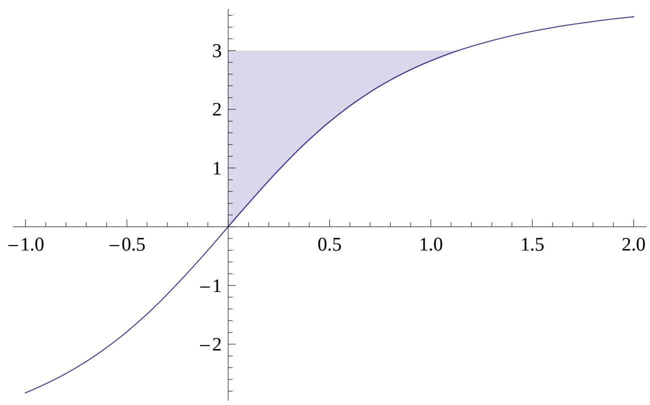
$$\begin{aligned}
\text{Area} &= \int_0^4 \left((\sqrt{x} + 3) - \left(\frac{1}{2}x + 3 \right) \right) \, dx \\
&= \int_0^4 \left(\sqrt{x} - \frac{1}{2}x \right) \, dx \\
&= \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2 \right]_0^4 \\
&= \frac{2}{3} \cdot 4^{3/2} - \frac{1}{4} \cdot 4^2 \\
&= \frac{4}{3}.
\end{aligned}$$

Problem 28

Problem. Sketch the region bounded by the graphs of the equations and find the area of the region.

$$f(y) = \frac{y}{\sqrt{16 - y^2}},$$
$$g(y) = 0,$$
$$y = 3$$

Solution. If we sketch the graph, we see which is the upper function.



It would be simpler to integrate along the y-axis.

$$\text{Area} = \int_0^3 \frac{y}{\sqrt{16 - y^2}}\, dy.$$

Let $u = 16 - y^2$ and $du = -2y\, dy$. Then

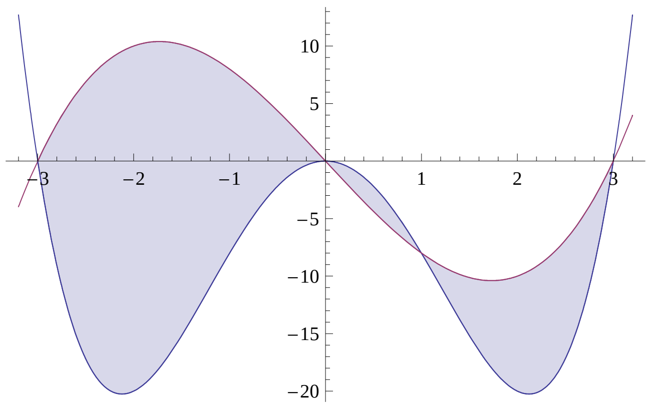
$$\begin{aligned}
\text{Area} &= -\frac{1}{2} \int_0^3 \frac{(-2y)}{\sqrt{16 - y^2}}\, dy \\
&= -\frac{1}{2} \int_{16}^7 \frac{1}{\sqrt{u}}\, du \\
&= -\frac{1}{2} \left[2\sqrt{u}\right]_{16}^7 \\
&= -\left(\sqrt{7} - \sqrt{16}\right) \\
&= 4 - \sqrt{7}.
\end{aligned}$$

Problem 34

Problem. Find the area of the region bounded by the graphs of the equations.

$$f(x) = x^4 - 9x^2,$$
$$g(x) = x^3 - 9x$$

Solution. If we sketch the graph, we see which is the upper function.



We need to find the points of intersection.

$$\begin{aligned} x^4 - 9x^2 &= x^3 - 9x, \\ x^4 - x^3 - 9x^2 + 9x &= 0, \\ x(x^3 - x^2 - 9x + 9) &= 0, \\ x(x^2 - 9)(x - 1) &= 0, \\ x(x - 3)(x + 3)(x - 1) &= 0. \end{aligned}$$

There are four points of intersection: $(-3, 0)$, $(0, 0)$, $(1, -8)$, and $(3, 0)$. We need three integrals.

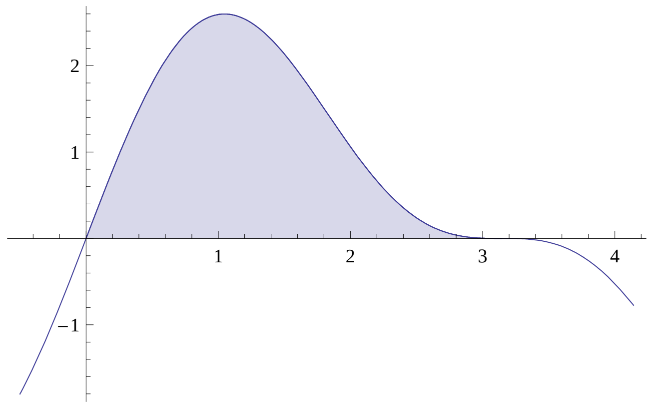
$$\begin{aligned}
\text{Area} &= \int_{-3}^{0} \big((x^3 - 9x) - (x^4 - 9x^2)\big)\ dx + \int_{0}^{1} \big((x^4 - 9x^2) - (x^3 - 9x)\big)\ dx \\
&\qquad + \int_{1}^{3} \big((x^3 - 9x) - (x^4 - 9x^2)\big)\ dx \\
&= \int_{-3}^{0} (-x^4 + x^3 + 9x^2 - 9x)\ dx + \int_{0}^{1} (x^4 - x^3 - 9x^2 + 9x)\ dx \\
&\qquad + \int_{1}^{3} (-x^4 + x^3 + 9x^2 - 9x)\ dx \\
&= \left[-\frac{1}{5}x^5 + \frac{1}{4}x^4 + 3x^3 - \frac{9}{2}x^2\right]_{-3}^{0} + \left[\frac{1}{5}x^5 - \frac{1}{4}x^4 - 3x^3 + \frac{9}{2}x^2\right]_{0}^{1} \\
&\qquad + \left[-\frac{1}{5}x^5 + \frac{1}{4}x^4 + 3x^3 - \frac{9}{2}x^2\right]_{1}^{3} \\
&= -\left(-\frac{1}{5}(-3)^5 + \frac{1}{4}(-3)^4 + 3(-3)^3 - \frac{9}{2}(-3)^2\right) + \left(\frac{1}{5} - \frac{1}{4} - 3 + \frac{9}{2}\right) \\
&\qquad + \left(-\frac{1}{5}\cdot 3^5 + \frac{1}{4}\cdot 3^4 + 3\cdot 3^3 - \frac{9}{2}\cdot 3^2\right) - \left(-\frac{1}{5} + \frac{1}{4} + 3 - \frac{9}{2}\right) \\
&= 67.7.
\end{aligned}$$

Problem 43

Problem. Find the area of the region bounded by the graphs of the equations.

$$\begin{aligned}
f(x) &= 2\sin x + \sin 2x, \\
y &= 0, \\
0 &\le x \le \pi
\end{aligned}$$

Solution. If we sketch the graph, we see which is the upper function.



$$\begin{aligned}
\text{Area} &= \int_0^\pi (2\sin x + \sin 2x)\, dx \\
&= \left[-2\cos x - \frac{1}{2}\cos 2x\right]_0^\pi \\
&= \left(-2\cos\pi - \frac{1}{2}\cos 2\pi\right) - \left(-2\cos 0 - \frac{1}{2}\cos 0\right) \\
&= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) \\
&= 4.
\end{aligned}$$

Problem 80

Problem. The surface of a machine part is the region between the graphs of $y_1 = |x|$ and $y_2 = 0.08x^2 + k$ (see figure).

(a) Find k where the parabola is tangent to the graph of y_1.

(b) Find the area of the surface of the machine part.

Solution. (a) One way to find k is to first find where the parabola intersects the line $y = x$. For very small k, there will be two intersection points. For large k, there will be no intersection point. The quadratic formula will make that clear when we solve the equation.

$$\begin{aligned}
0.08x^2 + k &= x, \\
0.08x^2 - x + k &= 0, \\
x &= \frac{1 \pm \sqrt{1 - 0.32k}}{0.16}.
\end{aligned}$$

Now it is clear that there will be exactly one intersection point (for positive x) when $1 - 0.32k = 0$. We solve that and get $k = 3.125$.

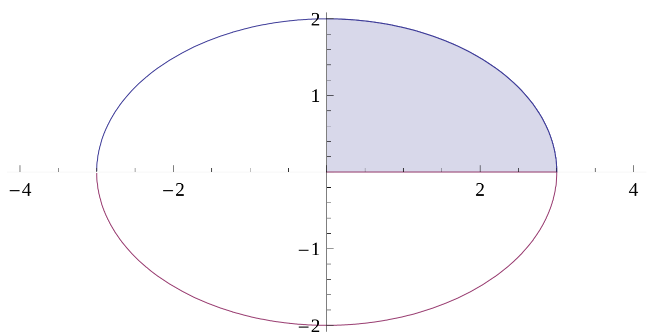
(b) Now the upper function is $y = 0.08x^2 + 3.125$ and the lower function is $y = x$. The intersection point itself is at $x = \frac{1}{0.16} = 6.25$ (from the quadratic equation in part (a)).

$$\begin{aligned}\text{Area} &= 2\int_0^{6.25} \left((0.08x^2 + 3.125) - x\right)\, dx \\ &= 2\left[\frac{0.08}{3}x^3 + 3.125x - \frac{1}{2}x^2\right]_0^{6.25} \\ &= 2\left(\frac{0.08}{3}\cdot 6.25^3 + 3.125\cdot 6.25 - \frac{1}{2}\cdot 6.25^2\right) \\ &= 13.0208.\end{aligned}$$

Problem 82

Problem. Let $a > 0$ and $b > 0$. Show that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

Solution. The graph is



with only the first quadrant shaded. We may find the area in the first quadrant and multiply by 4.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

may be written as the function

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

in the first quadrant.

$$\text{Area} = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2}\, dx.$$

At this point, we do not know how to find an antiderivative of $\sqrt{a^2 - x^2}$. However, watch what happens when we make the substitution $x = au$ and $dx = a\, du$.

$$\begin{aligned} \text{Area} &= \frac{4b}{a} \int_0^1 \sqrt{a^2 - a^2u^2} \cdot a\, du \\ &= 4ab \int_0^1 \sqrt{1 - u^2}\, du. \end{aligned}$$

The integral

$$\int_0^1 \sqrt{1 - u^2}\, du$$

represents the area of one-quarter of a unit circle, whose area we know to be $\frac{\pi}{4}$. Therefore,

$$\begin{aligned} \text{Area} &= 4ab \cdot \frac{\pi}{4} \\ &= \pi ab. \end{aligned}$$